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Equilibrium Pure States and Nonequilibrium Chaos

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We consider nonequilibrium systems such as the Edwards–Anderson Ising spin glass at a temperature where, in equilibrium, there are presumed to be (two or many) broken-symmetry pure states. Following a deep quench, we argue that as time $t \to \infty$, although the system is usually in some pure state locally, either it *never* settles permanently on a fixed length scale into a single pure state, or it does, but then the pure state depends on *both* the initial spin configuration *and* the realization of the stochastic dynamics. But this latter case can occur only if there exists an uncountable number of pure states (for each coupling realization) with almost every pair having zero overlap. In both cases, almost no initial spin configuration is in the basin of attraction of a single pure state; that is, the configuration space (resulting from a deep quench) is all boundary (except for a set of measure zero). We prove that the former case holds for deeply quenched 2D ferromagnets. Our results raise the possibility that even if more than one pure state exists for an infinite system, time averages do not necessarily disagree with Boltzmann averages.

KEY WORDS: Spin glass; nonequilibrium dynamics; deep quench; stochastic Ising model; broken ergodicity; coarsening; persistence; damage spreading.

1. INTRODUCTION

In this paper we study nonequilibrium dynamics of spin systems. While our approach is general, and covers both ordered and disordered, Ising and non-Ising systems, in the presence or absence of a magnetic field, we will for specificity here focus mostly on the dynamics of the Edwards–Anderson[(12)] Ising spin glass in zero field. We make no assumptions about the real- or state-space structure of the low-temperature spin glass phase,

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but instead derive several general principles and then explore their consequences.

Our results indicate that equilibrium pure state structure plays an important role in nonequilibrium dynamics. E.g., we will show that (at fixed temperature) a system with many pure states may have very different dynamical behavior than one with only a single pair[3]—so nonequilibrium dynamics can serve as an important probe of the equilibrium pure state structure.

That the number of and relationships among pure states can affect nonequilibrium dynamics may seem surprising in light of a general supposition (see, for example, refs. 2 and 29 and many of the references therein) that the time evolution of an infinite system is confined within a single pure state for all finite times. A consequence of this supposition is the prevalent notion that, if there exists at some temperature T more than one pure state (e.g., due to broken symmetry), then necessarily the limits $t \to \infty$ (time) and $N \to \infty$ (system size) do not commute (when, say, measuring the state of some observable). An equivalent statement is that time averages (as performed in the lab) and Boltzmann averages (as performed on paper or the computer) will give differing results. (One proposal[(23)] for avoiding this problem in theoretical treatments of spin glass dynamics is to start with a Boltzmann distribution of initial configurations, rather than a single one.) We will argue, however, that these are not necessary consequences of broken symmetry or multiple pure states; they may be true in some cases, but not in general. We examine here in which contexts they are true and in which they are not.[4] We begin by constructing dynamical probability measures on spin configurations, which will enable us to clarify notions such as time evolution within a pure state.

2. DYNAMICAL MEASURES

We will mostly, but not exclusively, consider the EA Ising spin glass in zero field. Its Hamiltonian is given by:

$$\mathscr{H}_{\mathscr{J}} = -\sum_{\langle xy \rangle} J_{xy} \sigma_x \sigma_y \tag{1}$$

[3] We emphasize that we are talking here about the equilibrium pure states present *at* the temperature at which the dynamics is observed, as distinguished from the proposal (see, e.g., ref. 22) that metastable states with $O(1)$ free energy barriers affecting dynamics at a given temperature are the precursors of equilibrium pure states at lower temperatures.

[4] Except in some comments about other papers, we will not refer to metastable states, often proposed as responsible for the anomalous dynamical behavior of spin glasses. While we have no argument with this, we question the usefulness of the usual practice of inserting metastability by hand, requiring a guess as to the structure (usually in state space) and nature of the metastable states. In the treatment presented here, metastability may emerge naturally.

where the sites $x, y \in \mathbf{Z}^d$ and the sum is taken over nearest neighbors only. The couplings J_{xy} are independent random variables, whose common probability density is symmetric about zero; we let $\mathcal{J}$ denote a particular realization of the couplings.

We consider Glauber dynamics of an *infinite* EA Ising spin glass starting from an initial (infinite-volume) spin configuration σ^0. We regard σ^0 as chosen from the (infinite temperature) ensemble where the individual spins are independent random variables equally likely to be $+1$ or -1. This corresponds to the experimental situation following a deep quench, and also (for smaller systems) most numerical simulations. We denote by ω a given realization of the dynamics. So there are three sources of randomness in the problem, with realizations given by $\mathcal{J}$, σ^0, and ω. All three are needed to determine the spin configuration at time t (we take the starting time to be 0). From here on we take $\mathcal{J}$ to be fixed. We note that the exact choice of spin flip rates plays no role in our analysis at positive temperature, as long as detailed balance is satisfied. At $T=0$ we consider only the widely used dynamical rule where flips that are energy lowering (or neutral or raising) occur at rate 1 (or 1/2 or 0). In all cases, ω can be regarded in the usual way as a collection of random times $(t_{x,i}: x \in \mathbf{Z}^d, i=1,2,...)$ when spin flips are considered (forming a Poisson process for each x) along with uniformly distributed random numbers $u_{x,i}$ that determine if the flips are taken.

We now define a dynamical probability measure $\nu_{t^*, \tau(t^*)}$ on the spin configurations, for $0 \leqslant \tau(t^*) \leqslant t^*$. Given some σ^0, we let the system evolve according to some ω up to a time $t^* - \tau(t^*)$, after which we average over the dynamics up to time t^*. That is, if we denote the dynamical (Markov) process as $\sigma^t = \sigma^t(\omega)$ for $t \geqslant 0$, then $\nu_{t^*, \tau(t^*)}$ is the conditional distribution of σ^{t^*} conditioned on ($\mathcal{J}$ and σ^0 and) all $(t_{x,i}, u_{x,i})$'s with $t_{x,i} \leqslant t^* - \tau(t^*)$. So $\nu_{t^*} \equiv \nu_{t^*, t^*}$ represents a complete averaging over the dynamics (and corresponds to the usual dynamical measure—i.e., to the distribution of σ^{t^*} for fixed $\mathcal{J}$ and σ^0), while $\nu_{t^*, 0}$ represents no averaging at all (and hence a single spin configuration). To avoid awkward notation, we generally suppress the dependence, which is understood, of ν on $\mathcal{J}$, σ^0, ω (up to time $t^* - \tau$), and T. We also note that neither t^* nor τ depend on $\mathcal{J}$, σ^0 or ω. In Section 5 we will briefly discuss the construction of measures based instead on time averaging for *fixed* dynamics.

We can now begin to answer the question, what does it mean for the system to evolve or settle into (or "spend all its time inside") a single pure state? Consider the cube Λ_L of linear size L and volume $|\Lambda_L|$ (which may be arbitrarily large) centered at the origin. When $T > 0$ (but not at $T=0$), we expect that for sufficiently large t^* (depending on L) and for almost every σ^0, the measure ν_{t^*} approximates a (possibly mixed, possibly

t^*-dependent) Gibbs state *restricted to the cube* Λ_L. By this we mean that there is some (infinite volume) Gibbs state ρ_{t^*} such that for any L and any spin configuration $\sigma^{(L)}$ in Λ_L, the probability assigned to $\sigma^{(L)}$ by the dynamical measure ν_{t^*} and that assigned by the equilibrium Gibbs measure ρ_{t^*} approach each other as $t^* \to \infty$; that is, $\nu_{t^*}(\sigma^{(L)}) - \rho_{t^*}(\sigma^{(L)}) \to 0$ as $t^* \to \infty$. More generally, we expect that for almost every σ^0 and ω, $\nu_{t^*, \tau(t^*)}$ approximates some Gibbs state $\rho_{t^*, \tau(t^*)}$ providing only that $\tau(t^*) \to \infty$.

The notion that, as t^* increases, $\nu_{t^*, \tau(t^*)}$ is increasingly well approximated by some infinite volume Gibbs state (possibly depending on t^*), may seem surprising—especially in view of frequent assertions (see, e.g., ref. 9) that equilibrium states are of little relevance for the nonequilibrium dynamics of infinite systems. In fact, this notion is nothing more than the property that for any L and any two spin configurations $\sigma^{(L)}$, $\sigma'^{(L)}$, defined within Λ_L and agreeing on its (internal) boundary, the ratio $\nu_{t^*, \tau}(\sigma^{(L)})/\nu_{t^*, \tau}(\sigma'^{(L)})$ converges to the usual *finite* volume Gibbs expression coming from the EA Hamiltonian. This kind of convergence may be expected as it is similar to the conjectured property of Glauber dynamics that only Gibbs states are stationary.

To answer the question posed above about the meaning of settling into a single pure state, we now note that if $\tau(t^*)$ also grows sufficiently slowly, then $\nu_{t^*, \tau(t^*)}$ should[5] (for most ω's) approximate a *pure* (i.e., extremal) Gibbs state $\rho^{\alpha(t^*)}$ depending on σ^0 and ω (up to time $t^* - \tau$). If on every fixed (and arbitrarily large) lengthscale L this ρ^α eventually becomes independent of time (after a timescale depending on L), then the system has settled into the pure state α.

We now present our main results. Unless otherwise indicated, all are for $T > 0$ and are independent of space dimension. Our first result concerns the fully averaged dynamical measure ν_{t^*}.

3. EVOLUTION OF THE DYNAMICAL MEASURE

Theorem 1. Given some $\mathcal{J}$, assume that for almost every σ^0, ν_{t^*} converges to a limiting *pure* Gibbs state ν_∞ as $t^* \to \infty$. Then ν_∞ is the same pure state for almost every σ^0.

[5] The tracking of $\nu_{t^*, \tau(t^*)}$ by the *pure* state $\rho^{\alpha(t^*)}$ is in general weaker than the tracking by $\rho_{t^*, \tau(t^*)}$ in that $\alpha(t^*)$ may not have a limit and $\nu_{t^*, \tau(t^*)}(\sigma^{(L)}) - \rho^{\alpha(t^*)}(\sigma^{(L)})$ may tend to zero only in probability rather than for almost all ω's. We do expect, however, that for almost all ω's, $\rho^{\alpha(t^*)}$ tracks $\nu_{t^*, \tau(t^*)}$ for *most* large t^*'s (i.e., when domain walls are far from the origin).

Proof. We use a coupling argument where a single ω is used with two starting spin realizations σ^0 and σ'^0 that differ at only finitely many sites. Then, by the nature of Glauber dynamics, there is a positive probability (with respect to the ω's) that the two spin configurations merge after a finite time. Let $\theta > 0$ represent the probability that the difference in configurations disappears by time t_0. So for $t^* \geqslant t_0$,

$$\nu_{t^*}^{\sigma^0} = \theta\mu_{t^*} + (1-\theta)\,\tilde{\mu}_{t^*}, \qquad \nu_{t^*}^{\sigma'^0} = \theta\mu_{t^*} + (1-\theta)\,\tilde{\mu}'_{t^*} \tag{2}$$

where $\mu_{t^*}, \tilde{\mu}_{t^*}$, and $\tilde{\mu}'_{t^*}$ are some probability measures. It follows that for A any (measurable) set of spin configurations,

$$|\nu_{t^*}^{\sigma^0}(A) - \nu_{t^*}^{\sigma'^0}(A)| \leqslant 1-\theta \tag{3}$$

The same is true with t^* replaced by ∞, first for A a locally defined event and then, by approximation, for general A. The strict positivity of θ then implies that $\nu_\infty^{\sigma^0}$ and $\nu_\infty^{\sigma'^0}$ cannot be mutually singular measures (i.e., living on completely disjoint regions of configuration space) and hence,[20] as pure states, they must be identical. So a change of finitely many spins in σ^0 doesn't change $\nu_\infty^{\sigma^0}$, and we can then use the Kolmogorov zero-one law[14] to conclude that the final pure state must be independent of σ^0. ∎

Despite the innocuous look of the theorem, it has important consequences. Its conclusion is obvious if there exists only one pure state, but it applies equally to the situation where many pure states exist. Of course, it could logically be that only one pure state is "present" (in the sense that the conclusion of the theorem is valid) while other pure states exist but are not physically relevant in our dynamical context. (This might even be the case, for example, in the EA spin glass with a small nonzero field.) But, if the conclusion of the theorem is not valid, then only one of two possibilities can occur:[6] either (1) ν_∞ is a mixed Gibbs state (which may or may not depend on σ^0), or (2) ν_{t^*} does not converge as $t^* \to \infty$. We note that the latter case is already known to occur[17] in some $1D$ disordered ferromagnets at $T=0$; on the other hand, if ν_∞ exists and does not depend on σ^0, then it would be analogous to (and perhaps the same as) $\rho_{\mathscr{J}}$, the average over the metastate discussed in ref. 32.

Let's consider each of these two possibilities, taking into account that ν_{t^*} is the average over ω's of $\nu_{t^*,\,\tau(t^*)}$. Possibility (1) implies that although $\nu_{t^*,\,\tau(t^*)}$ (with properly chosen τ) is approximately a pure state $\rho^{\alpha(t^*)}$, that

[6] Logically, there is a third possibility that ν_∞ is not a Gibbs state, but this would violate the expected behavior discussed in Section 2 and hence we disregard it (except when $T=0$—see Section 4 and ref. 30).

pure state depends not only on σ^0 (as expected) but also on ω. This allows (but doesn't require—see Subsection 4.1) the system always to "land" in a pure state in the sense that $\nu_{t^*, \tau(t^*)} \to \rho^{\alpha(\sigma^0, \omega)}$ —but then the pure state is (almost) never determined solely by σ^0.

Now, the basin of attraction of a pure state $\bar{\alpha}$ may be defined as the set of σ^0's such that $\alpha(\sigma^0, \omega) = \bar{\alpha}$ for almost every ω (see ref. 13 for related discussions). We claim that if σ^0 is in the basin of attraction of some pure state, then by a modified version (see below) of the coupling argument used in the proof of Theorem 1, the same will be true after a change of finitely many spins in σ^0 and so, by the Kolmogorov zero-one law, the set of σ^0's that are in some basin of attraction has probability either zero or one. Therefore, if many pure states are present (i.e., if $\alpha(\sigma^0, \omega)$ is not the same for almost all σ^0 and ω), then the *union of all their basins of attraction must form a set of measure zero in the space of* σ^0's; i.e., the configuration space resulting from a deep quench is all "boundary" in the sense that almost every initial configuration will land in one of several (or many) pure states depending on the realization of the dynamics (if it lands at all).

The modified coupling argument is as follows. Let σ^0 and σ'^0 be fixed and let D denote the finite set of x's where they differ. Rather than using the same ω for the coupled processes, we take an $\omega = (t_{x,i}, u_{x,i})$ and an $\omega' = (t'_{x,i}, u'_{x,i})$ that are identical for x outside D but for x inside D, they are identical only for times after σ^t and σ'^t merge. For earlier times, ω and ω' inside D are independent of each other. If A' is an event defined in terms only of ω' with $\text{Prob}(A') > 0$ and M denotes the event of eventual merger of the two processes, then since ω inside D may (with small but strictly positive probability) force a merger by a small time ε, it follows that $\text{Prob}(A' \cap M) > 0$. Assuming $\alpha(\sigma^0, \omega) = \bar{\alpha}(\sigma^0)$ for almost every ω, we take A' to be the complement of the event that $\alpha(\sigma'^0, \omega') = \bar{\alpha}(\sigma^0)$ so that $\text{Prob}(A' \cap M) > 0$ would yield the contradiction that $\text{Prob}(\alpha(\sigma^0, \omega) = \bar{\alpha}(\sigma^0)) < 1$. It follows that $\text{Prob}(A') = 0$ and so $\alpha(\sigma'^0, \omega') = \bar{\alpha}(\sigma^0)$ for almost every ω', which is exactly the claim made in the previous paragraph.

It has been speculated[27] that slow relaxation in spin glasses may be due to points in (high-dimensional) state space always being "near" a boundary. What we've shown here differs in fundamental respects: our conclusion is that almost every point in state space is actually *on* a boundary, and therefore the dynamical consequences are not restricted to very low temperatures.

We note finally that Theorem 1 may be relevant to damage spreading,[1, 21, 24] where one asks whether the damage (i.e., discrepancy) between σ^t and σ'^t (with a single ω) grows as $t \to \infty$. Theorem 1 suggests that if damage spreading occurs, then ν_{t^*} doesn't converge to a single pure state (e.g., it might converge to a mixed state, as above).

4. LOCAL NON-EQUILIBRATION

Before discussing possibility (2), let us consider the physical picture implied by Theorem 1. Roughly speaking, some time after an initial quench the system will form domains, whose average size increases with time, corresponding to the different pure states. This scenario has been analyzed for the two-state droplet picture.[15, 25, 34] It is also a well-known scenario for coarsening in a ferromagnet following a deep quench.[6] (Of course, in contrast to the spin glass case, one *does* know how to prepare a ferromagnet in a pure state; for a general discussion, see ref. 33.)

In ref. 6 it was stated that the infinite homogeneous ferromagnet never reaches equilibrium in any finite time (following a deep quench) because the domain sizes (in this case, of positive and negative magnetization) increase with time but are never infinite on any finite timescale. We do not consider this by itself to be nonequilibration because it does not preclude the possibility that on any *finite* lengthscale, the system equilibrates after some finite time, in the sense that after that time domain walls cease to move across the region. Instead, we now propose a much stronger version of nonequilibration—the possibility of *local non-equilibration* (LNE) on any *finite* lengthscale, which is implied by possibility (2) and could also occur with possibility (1). We will discuss the difference between these two cases of LNE in Subsection 4.1, but for now will not distinguish between the two. We will also discuss below the relation between LNE and persistence exponents.[7, 10, 11, 28]

By LNE we mean that in any fixed finite region, the system never settles down into a pure state. That is, domain walls do not simply move farther from the region as time progresses, but continually return and sweep across it, changing the state within. More precisely, LNE is said to occur unless there is some choice of $\tau(t^*)$ such that for almost all σ^0 and ω, $\nu_{t^*, \tau(t^*)} \to \rho^{\alpha(\sigma^0, \omega)}$, a *pure* state. If LNE occurs, it would force us to revise the usual dynamical definition,[2] of, e.g., the EA order parameter. It could also mean that, for infinite systems, time averages and Gibbs averages could agree, despite the presence of many pure states. We will return to this in Section 5 after investigating LNE in more detail by means of the next theorem, which applies to both homogeneous and disordered systems.

Theorem 2. If only a single pair (or countably many, including a countable infinity) of pure states exists (with fixed $\mathscr{J}$) and these all have nonzero EA order parameter, then LNE occurs.

Proof. Suppose that there exists at T (and for the given $\mathscr{J}$) only a single pair of pure states, and assume that LNE does *not* occur so that for each σ^0 and ω, there is a limiting pure state $\alpha(\sigma^0, \omega)$. The overlap of $\alpha(\sigma^0, \omega)$ and $\alpha(\sigma'^0, \omega')$ is

$$Q(\mathscr{J}, \sigma^0, \omega, \sigma'^0, \omega') = \lim_{L \to \infty} |\Lambda_L|^{-1} \sum_{x \in \Lambda_L} \langle \sigma_x \rangle_{\alpha(\sigma^0, \omega)} \langle \sigma_x \rangle_{\alpha(\sigma'^0, \omega')} \tag{4}$$

where $\langle \cdot \rangle_\alpha$ denotes the (thermal) average with respect to the pure state ρ^α. The possible overlap values can be only $\pm q_{\mathrm{EA}}$ and both of these outcomes must have positive probability of occurring (as σ^0, ω, σ'^0, ω' vary independently to yield a pair of replicas). But the translation-ergodicity of each of the distributions from which $\mathscr{J}$, σ^0, ω, σ'^0, ω' are chosen implies the same for the joint (product) distribution of $(\mathscr{J}, \sigma^0, \omega, \sigma'^0, \omega')$. Thus, the fact that the overlap Q is a translation-invariant function implies that Q must be constant for almost all realizations, leading to a contradiction. This argument can be immediately extended to any countable number of pure state pairs. ∎

This proof also shows that if LNE does not occur (and the limiting pure states $\alpha(\sigma^0, \omega)$ have nonzero q_{EA}), then almost every (as σ^0, ω, σ'^0, ω' vary) overlap of the pair of pure states, $\alpha(\sigma^0, \omega)$ and $\alpha(\sigma'^0, \omega')$, is zero. This leads to:

Corollary. If LNE does not occur and $q_{\mathrm{EA}} \neq 0$, then there must be an *uncountable* number of pure states, with almost every pair (in the above sense) having overlap zero.

This shows that, as stated in the introduction, nonequilibrium dynamics provides important information on the structure of equilibrium pure states. It also suggests a dynamical test of the two-state picture: search for chaotic time dependence in $v_{t^*, \tau(t^*)}$. If LNE does not occur, then the two-state picture has been ruled out: there must be an uncountable number of states, almost all of which have overlap zero (consistent with the results of ref. 32). If LNE does occur then neither the two-state nor the many-state pictures have been ruled out.

How might one go about observing LNE in a spin glass, where, unlike the ferromagnet, one doesn't know what a domain wall looks like? Here one can use the clustering property that characterizes pure states in general. E.g., a truncated 2-point correlation function of the form $\langle \sigma_x \sigma_0 \rangle - \langle \sigma_x \rangle \langle \sigma_0 \rangle$ approaches 0 as $|x| \to \infty$ if the averaging is done in a pure state, but not otherwise. In the current context a pure state average corresponds to a dynamical average using $v_{t^*, \tau}$ with $\tau \ll t^*$. In principle one could evaluate this correlation function numerically for $t^* \gg \tau \gg |x| \gg 1$; if it does not approach zero for increasing values of these parameters, that would constitute a clear signal of the occurrence of LNE.

An important consequence of Theorem 2 is that LNE must also occur at positive temperature in the $2D$ uniform Ising ferromagnet and the random Ising ferromagnet for $d < 4$. (However, the argument for LNE in random ferromagnets is not entirely rigorous as there is no complete analysis of interface pure states there.[18] There is though a rigorous proof that for the SOS approximation, these states do exist for $d \geqslant 4$ and do not for $d < 4$; see refs. 4 and 5).

Moreover, in the $2D$ homogeneous ferromagnet (on the square lattice) results on LNE can be extended to $T = 0$, in the sense there that σ^t does not converge as $t \to \infty$; in fact, *every* spin flips infinitely often. The proof (by contradiction) is based on showing that if some spin (say at the origin) remained fixed forever, then (by translation ergodicity and spin flip symmetry) so would two spins of opposite sign on the x and y axes. But then there would be a domain wall passing through the rectangle determined by these three spins and "cutting off" one of them. In every time interval there would be a nonzero probability of the domain wall moving to flip that spin, and so with probability one, it eventually would flip. For the full proof, see ref. 30.

It is also shown in ref. 30 that for many systems σ^t *does* converge to some limit at $T = 0$. This is based on a very general argument that there can be only finitely many flips at any site that strictly lower the energy. Examples include spin glasses and random ferromagnets where the common distribution of the J_{xy}'s is continuous, and either has a finite mean (as in most ordinary models) or else is sufficiently "spread out," as in the highly disordered spin glass or ferromagnet.[31] Other examples are homogeneous ferromagnets (or homogeneous antiferromagnets or $\pm J$ spin glasses) on lattices with an odd number of nearest neighbors, such as the $2D$ hexagonal lattice or a double-layered $2D$ square lattice. (There are also systems, such as the $\pm J$ spin glass on the square lattice, where some spins flip only finitely many times and some spins flip infinitely often.[19]) In light of these results, we restrict the term LNE to $T > 0$, since in the zero-temperature situations where σ^t converges, the limit configuration is typically only metastable rather than a ground state and so equilibration has not really occurred. In these systems one can define a dynamical order parameter, related to the autocorrelation, that does *not* decay to zero.

To further clarify the discussion of LNE for the ferromagnet, we note that it is a phenomenon separate from the spontaneous formation (at positive temperature) of domains of, say, the minus phase within the plus phase. That is, on a timescale exponential in L, there will form a domain containing the origin (of size L) of the minus phase. Similarly, for a finite system of size L, the entire system will flip back and forth between the plus and minus phases on a timescale exponential in L. However, this is different from LNE, which presumably takes place on time scales of some power of L. Also, the spontaneous formation of droplets described above cannot occur at $T=0$, but as already discussed, in the $2D$ ferromagnet the phenomenon of domain walls forever sweeping across any finite region persists at zero temperature.

Since the existence of LNE for all $T<T_c$ in the 2D Ising ferromagnet may seem surprising, we present a possible physical mechanism for this case which may also shed light on LNE in general. The initial spin configuration has (with probability one) no infinite domains. As the configuration evolves, some domains shrink and others coalesce. So the origin should always be contained in a finite domain, whose size will usually be slowly decreasing, but sporadically will have a large change either by coalescing or because a domain wall passes through the origin and the identity of the domain changes. Thus LNE is primarily the result of nonequilibrium domain wall motion (driven by mean curvature) combined with the complex domain structure resulting from the original quench. It is also consistent with phase separation (as would be expected from equilibrium roughening arguments). In particular, the occurrence of LNE does not preclude the divergence with t of the *mean* scale of the domain containing the origin (although for *fixed* σ^0 and ω, there is a much more complex behavior, as indicated above).

4.1. LNE and Chaotic Time Dependence

We noted earlier in this section that LNE can occur in the context of either possibility (1) (the fully averaged dynamical measure ν_{t*} has a limit, which is a mixed state) or (2) (ν_{t*} does not converge). That is, LNE *must* occur if possibility (2) holds, but may or may not occur if possibility (1) holds. We now explore further the distinctions between the two cases of LNE.

As described earlier, one way for possibility (1) to occur is if, for some choice of $\tau(t^*)$, $\nu_{t^*,\,\tau(t^*)} \to \rho^{\alpha(\sigma^0,\,\omega)}$ for almost all σ^0 and ω, where $\rho^{\alpha(\sigma^0,\,\omega)}$ is some pure state. But it could also happen that for any choice of $\tau(t^*) \ll t^*$, $\nu_{t^*,\,\tau(t^*)}$ never settles down to a single pure state—so the system is usually in a pure state locally, but the pure state forever changes. Nevertheless, a full average over the dynamics (i.e., letting $\tau = t^*$) still yields a single limit. This is to be contrasted with possibility (2), where even the fully averaged measure never settles down.

To clarify these statements, we use the illustration of the $2D$ homogeneous ferromagnet below T_c, where we know LNE to occur by Theorem 2. Suppose that possibility (1) occurs. Then (for fixed σ^0 and large time) for approximately half of the dynamical realizations, a region of fixed lengthscale L surrounding the origin is in the up state (i.e., the pure Gibbs state ρ^+), and for most of the other half the same region is in the down state (the pure Gibbs state ρ^-), and this one-to-one ratio remains essentially fixed after some timescale depending on L. Then as $t^* \to \infty$, $v_{t^*} \to \bar{\rho}$, where $\bar{\rho}$ is the mixed Gibbs state $(1/2)\,\rho^+ + (1/2)\,\rho^-$. Nevertheless, in any given dynamical realization (with averaging done as usual after time $t^* - \tau$, with $1 \ll \tau \ll t^*$), the region never settles permanently into either ρ^+ or ρ^-.

By contrast, if possibility (2) occurs, then even the fully averaged dynamical measure v_{t^*} forever changes. This could happen (again for fixed σ^0) if the random dynamics fails to sufficiently "mix" the states (in which case one has, given σ^0, some amount of predictive power for determining from σ^0 the likely state of the system in the region for arbitrarily large times t^*). This is conceivable because even though σ^0 is globally unbiased between the plus and minus states, it does have fluctuations in favor of one or the other state of order $\sqrt{(L^*)^2}$ on lengthscale L^*; with L^* taken as an appropriate power of t^*, these fluctuations could (partially) predict the sign of the phase at the origin at time t^*. In possibility (1) on the other hand, there is a greater capability of the random dynamics to "mix" the states which eventually destroys the predictive power contained in the fluctuations of the initial state.

So there are really two kinds of non-equilibration, corresponding either to LNE in the framework of possibility (1) ("weak LNE") or else to LNE resulting from the stronger possibility (2). Because v_{t^*} evolves deterministically according to an appropriate master equation, its lack of a limit in possibility (2) corresponds to the usual notion of deterministic chaos and can thus legitimately be called chaotic time dependence (CTD).[17] If weak LNE occurs, this term is not appropriate because here the effect is due to the random dynamics.

In our discussion of LNE in the previous section, we do not yet know which of the cases correspond to CTD and which to weak LNE. This remains a problem for future investigation. However, we note that the occurrence of LNE (but not CTD) in homogeneous ferromagnets (on $\mathbf{Z}^d$) is implicit in the (nonrigorous for $d \neq 1$) analysis of persistence exponents[7, 10, 11, 28] in the sense that the fraction of sites that remain in the same phase from time t_1 to time t_2 tends to zero for $1 \ll t_1 \ll t_2$. On the other hand, the fact mentioned above that the $T = 0$ analogue of this phenomenon is *lattice dependent* appears to have gone unnoticed.

5. TIME-AVERAGED DYNAMICAL MEASURES

We return to one issue that needs discussion, and will be treated in greater detail elsewhere. If LNE occurs (e.g., because only a single pair of pure states is present), would a (long) time average of, say, the spin at the origin (or, for the ferromagnet, the magnetization in a finite region), give zero? The answer is: not necessarily. It could be that, after long times, the system has spent roughly equal amounts of time in both states, in which case the usual time average[(2)] (or a discrete average over equally spaced times) *would* approach zero. But it could also happen that, after any long time, the system has spent significantly more of its life in one or the other state (which itself would change with the observational timescale). In other words (still using the example of a two-state system), at any long time the weights of the two states, as defined by a dynamical measure involving a fixed ω and an average over uniformly spaced times, could be different from 1/2, and moreover they may change with time. This is analogous to an equilibrium phenomenon discovered by Külske[(26)] and seems to be exactly what occurs in homogeneous ferromagnets, as reported in recent numerical studies.[(11)] To get a zero average in this situation one would need to average over a sparse sequence of increasingly separated times.

6. SUMMARY

We have presented a rigorous approach to the dynamics of infinite spin systems which introduced various dynamical measures on the spin configurations, and considered whether they evolve into pure states. We showed that in the case of the EA Ising spin glass with broken spin-flip symmetry, one (or both) of two interesting things must happen: either LNE occurs, where the system never settles into any pure state (i.e., domain walls forever pass through any finite region, causing it forever to change its pure state),[7] or else there exist uncountably many pure states, with almost every pair having zero overlap.

We proved that the union of the basins of attraction of all pure states (again, if broken symmetry occurs) forms a set of measure zero in configuration space; i.e., almost every starting configuration is on a boundary between (several or possibly all) pure states. While this is also true for the ferromagnet, it obviously is still easy to prepare that system in a pure state.

[7] LNE may seem somewhat analogous to "weak ergodicity breaking" (see, e.g., ref. 3) in which the system forever moves to different metastable states with increasing lifetimes. There are important differences, however: that scenario explicitly requires the system to remain within a single pure state for all time (and unlike LNE is supposed to be able to occur even if there is only one pure state), and further makes no reference to what is occurring in real space.

But this result has serious dynamical consequences for the spin glass, and not only for deep quenches. Because of the possibility of chaotic temperature dependence,[8, 16] it is potentially relevant even for small temperature changes made slowly. Experimentally observed slow relaxation and long equilibration times in spin glasses may therefore be a consequence of small (relative to the system) domain size and slow (possibly due to pinning) motion of domain walls.

More generally, we have argued against a common viewpoint that pure state multiplicity is irrelevant to the dynamics of infinite (or very large) systems on finite timescales. A system need not—and in several cases, does not—spend all of its time in a single pure state, even locally. Because of this, it is also not necessarily true that "absolutely broken ergodicity"—i.e., the presence of more than one pure state separated by infinite barriers—implies that time averages and Boltzmann averages must disagree (or equivalently, that the limits $N \to \infty$ and $t \to \infty$ cannot commute). Both averages can be zero if for each state the (infinite) system locally spends (roughly) equal amounts of time in it and its global flip, as discussed in Section 5. The other possibility is that after almost any long time, the system has spent more of its life in one or the other state (which itself would change with the observational timescale). If this is the case, the averages should disagree, but due to a mechanism different from the standard one. Further development of these ideas, and a discussion of their application to experiment, will be presented in a future paper.

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